A Second Order Solution to the Lid-Driven Cavity Problem

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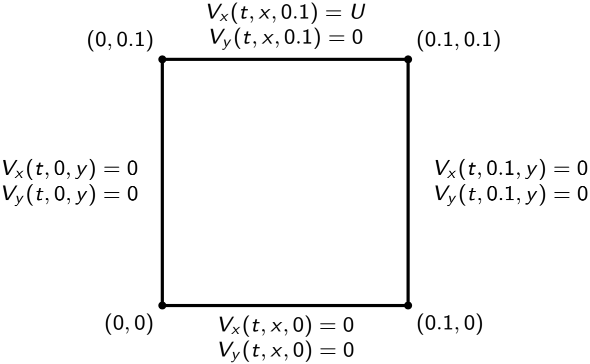
Problem Description



$$U = 0.5 \text{ m/s}$$
$$H = 0.1 \text{ m}$$
$$\nu = 0.001 \text{ m}^2/\text{s}$$

$$\text{Re} = \frac{UH}{\nu} = \frac{(0.5)(0.1)}{0.001} = 50$$

Navier-Stokes Formulation

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - \frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} = v\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

Stream Function Vorticity Formulation

$$\frac{\partial \zeta}{\partial t} = -u\frac{\partial \zeta}{\partial x} - v\frac{\partial \zeta}{\partial y} + \nu\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right)$$

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = -\zeta$$

A Numerical Step

1. Initialize velocity, vorticity, and stream-function fields
2. Compute the bounday conditions for vorticity using the stream-function field

$$\zeta_{\text{wall}} = -\frac{\partial^2 \Psi}{\partial \text{n}^2}\bigg|_{\text{wall}}$$

3. Solve the vorticity transport equation for the next time step
4. Solve the Poisson equation for the stream function
5. Compute the velocity field at new time step using

$$u = \frac{\partial \Psi}{\partial y} \quad v = -\frac{\partial \Psi}{\partial x}$$

6. Return to step 2 and repeat until convergence

[1]

[1] https://www.iist.ac.in/sites/default/files/people/psi-omega.pdf

The Numerical Scheme

- The Implicit Trapezoid method will be used for time advancement

$$\frac{\zeta^{n+1} - \zeta^{n}}{\Delta t} = \frac{1}{2}\left[\left.\frac{\mathrm{d}\zeta}{\mathrm{d}t}\right|_{n} + \left.\frac{\mathrm{d}\zeta}{\mathrm{d}t}\right|_{n+1} \right]$$

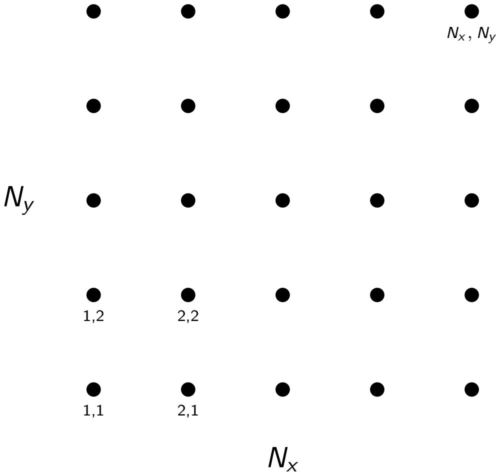
- 3-Point central differencing will be used for first and second derivatives

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{x_{i+1} - x_{i-1}}{2\Delta x} \qquad \frac{\partial^2 x}{\partial y^2} = \frac{x_{i+1} - 2x_i + x_{i-1}}{(\Delta x)^2}$$

- 3-Point one sided differences will be used for calculating $\zeta' s$ boundary condition

$$\zeta_{\mathrm{wall}} = a\Psi_i + b\Psi_{i+1} + c\Psi_{i+2}$$

Computational Mesh



Forward Difference

$$\frac{\partial^2 \Psi_{1,j}}{\partial x^2} = a\Psi_{1,j} + b\Psi_{2,j} + c\Psi_{3,j}$$

Term	$\Psi_{i,j}$	$\cancel{\frac{\partial \Psi_{i,j}}{\partial x}}$	$\frac{\partial^2 \Psi_{i,j}}{\partial x^2}$	$\frac{\partial^3 \Psi_{i,j}}{\partial x^3}$	$\frac{\partial^4 \Psi_{i,j}}{\partial x^4}$
$a\Psi_{1,j}$	a	0	0	0	0
$b\Psi_{2,j}$	b	bh	$b\frac{h^2}{2}$	$b\frac{h^3}{6}$	$b\frac{h^4}{24}$
$c\Psi_{3,j}$	c	$c2h$	$c2h^2$	$c\frac{4h^3}{3}$	$c\frac{2h^4}{3}$
$\frac{\partial^2 \Psi_{i,j}}{\partial x^2}$	0	0	1	0	0

$$\frac{\partial^2 \Psi_{1,j}}{\partial x^2} \approx \frac{-7\Psi_{1,j} + 8\Psi_{2,j} - \Psi_{3,j}}{2\Delta x^2} + \cancelto{0}{\frac{3v_{1,j}}{\Delta x}}$$

$$\zeta_{1,j} \approx \frac{7\Psi_{1,j} - 8\Psi_{2,j} + \Psi_{3,j}}{2\Delta x^2} + \ldots$$

$$\frac{\Delta x^2}{6}\frac{\partial^4 \Psi_{1,j}}{\partial x^4}$$

Backward Difference

$$\frac{\partial^2 \Psi_{i,N_y}}{\partial y^2} = a\Psi_{i,N_y} + b\Psi_{i,N_y-1} + c\Psi_{i,N_y-2}$$

Term	$\Psi_{i,j}$	$\cancel{\frac{\partial \Psi_{i,j}}{\partial y}}$	$\frac{\partial^2 \Psi_{i,j}}{\partial y^2}$	$\frac{\partial^3 \Psi_{i,j}}{\partial y^3}$	$\frac{\partial^4 \Psi_{i,j}}{\partial y^4}$
$a\Psi_{i,j}$	a	0	0	0	0
$b\Psi_{i,j-1}$	b	$-bh$	$b\frac{h^2}{2}$	$-b\frac{h^3}{6}$	$b\frac{h^4}{24}$
$c\Psi_{i,j-2}$	c	$-c2h$	$c2h^2$	$-c\frac{4h^3}{3}$	$c\frac{2h^4}{3}$
$\frac{\partial^2 \Psi_{i,j}}{\partial y^2}$	0	0	1	0	0

$$\frac{\partial^2 \Psi_{i,N_y}}{\partial y^2} \approx \frac{-7\Psi_{i,N_y} + 8\Psi_{i,N_y-1} - \Psi_{i,N_y-2}}{2\Delta y^2} - \cancelto{U}{\frac{3u_{i,N_y}}{\Delta y}}$$

$$\zeta_{i,N_y} \approx \frac{-7\Psi_{i,N_y} + 8\Psi_{i,N_y-1} - \Psi_{i,N_y-2}}{2\Delta y^2} - \frac{3U}{\Delta y} - \ldots$$

$$\frac{\Delta y^2}{6}\frac{\partial^4 \Psi_{i,N_y}}{\partial y^4}$$

Step 3: Time Step the Vorticity Transport Equation

Discretization of Vorticity Transport Equation

$$\frac{\partial \zeta_{i,j}}{\partial t} = \frac{u_{i,j}}{2h}\left(\zeta_{i-1,j} - \zeta_{i+1,j}\right) + \frac{v_{i,j}}{2h}\left(\zeta_{i,j-1} - \zeta_{i,j+1}\right) + \frac{\nu}{h^2}\left(\zeta_{i+1,j} + \zeta_{i-1,j} + \zeta_{i,j+1} + \zeta_{i,j-1} - 4\zeta_{i,j}\right)$$
$$= \underbrace{\left(\frac{u_{i,j}}{2h} + \frac{\nu}{h^2}\right)}_{a}\zeta_{i-1,j} + \underbrace{\left(-\frac{4\nu}{h^2}\right)}_{b}\zeta_{i,j} + \underbrace{\left(-\frac{u_{i,j}}{2h} + \frac{\nu}{h^2}\right)}_{c}\zeta_{i+1,j} + \underbrace{\left(\frac{v_{i,j}}{2h} + \frac{\nu}{h^2}\right)}_{d}\zeta_{i,j-1} + \ldots$$
$$\underbrace{\left(-\frac{v_{i,j}}{2h} + \frac{\nu}{h^2}\right)}_{e}\zeta_{i,j+1}.$$

For the unknown vector:

$$\{\zeta\} = \begin{Bmatrix} \zeta_{2,2} \\ \zeta_{3,2} \\ \vdots \\ \zeta_{N_x-1,2} \\ \vdots \\ \zeta_{2,N_y-1} \\ \zeta_{3,N_y-2} \\ \vdots \\ \zeta_{N_x-1,N_y-1} \end{Bmatrix}$$

$$\frac{\partial \zeta}{\partial t} = L\zeta + K$$

Step 3: Time Stepping the Vorticity Transport Equation

$$\frac{\partial \zeta}{\partial t} = L\zeta + K$$

$$L = \begin{bmatrix} B & C & & & \\ A & B & C & & \\ & \ddots & \ddots & \ddots & \\ & & A & B & C \\ & & & A & B \end{bmatrix}$$

$$A = \begin{bmatrix} d & & \\ & \ddots & \\ & & d \end{bmatrix} \qquad C = \begin{bmatrix} e & & \\ & \ddots & \\ & & d \end{bmatrix}$$

$$B = \begin{bmatrix} b & c & & & \\ a & b & c & & \\ & \ddots & \ddots & \ddots & \\ & & a & b & c \\ & & & a & b \end{bmatrix}$$

Step 3: What is K?

1. First Nx-2 Rows account for the left, right, and bottom walls
2. Last Nx-2 Rows account for the left, right, and top walls
3. Remaining rows account for left and right walls at interior points

Step 3: Time Stepping the Vorticity Transport Equation

$$\zeta^{n+1} = \zeta^n + \frac{\Delta t}{2}\left(L\zeta^n + L\zeta^{n+1} + 2K^n\right)$$

$$2\zeta^{n+1} = 2\zeta^n + \Delta t\left(L\zeta^{n+1} + L\zeta^n + 2K^n\right)$$

$$\zeta^{n+1}\left(2 - \Delta tL\right) = \zeta^n\left(2 + \Delta tL\right) + 2\Delta tK^2$$

$$\zeta^{n+1} = \frac{\zeta^n\left(2 + \Delta tL\right) + 2\Delta tK^n}{\left(2 - \Delta tL\right)}$$

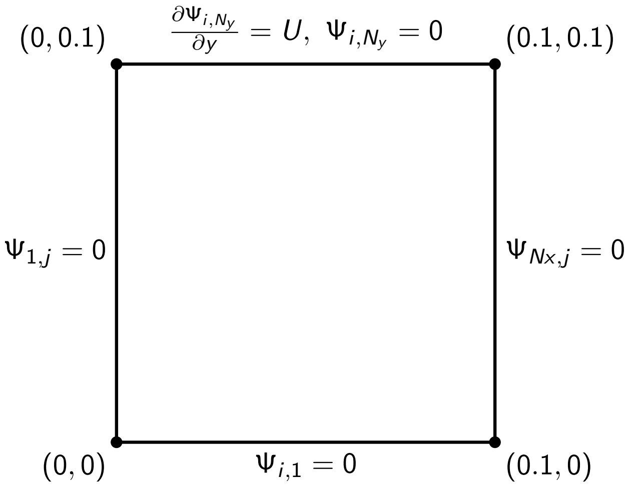
Pros

- zero phase magnitude error
- 2nd order accurate in time

Cons

- implict
- requires matrix "inversion" at each time step

Step 4: Solve Poisson's Equation for Ψ



Step 4: Solve Poisson's Equation for Ψ

Discretized Poisson's Equation

$$\frac{1}{h^2}\left(\Psi_{i+1,j} - 2\Psi_{i,j} + \Psi_{i-1,j}\right) + \frac{1}{h^2}\left(\Psi_{i,j+1} - 2\Psi_{i,j} + \Psi_{i,j-1}\right) = -\zeta_{i,j}$$

$$L = \begin{bmatrix} B & C & & & & \\ A & B & C & & & \\ & \ddots & \ddots & \ddots & & \\ & & A & B & C & \\ & & & A & B & C \\ & & & E_3 & E_2 & E_1 \end{bmatrix}$$

$$B = \frac{1}{h^2}\begin{bmatrix} -4 & 1 & & & \\ 1 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 1 \\ & & & 1 & -4 \end{bmatrix}$$

$$A = C = \frac{1}{h^2}\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

Residual Functions

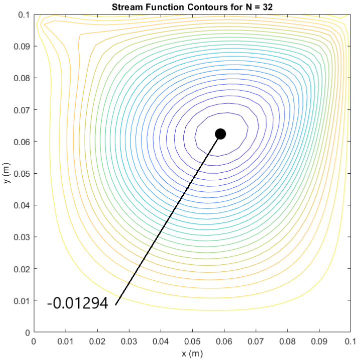
$$R_1 = \max\left(\frac{|\zeta^n_{i,j} - \zeta^{n-1}_{i,j}|}{\zeta^n_{i,j}}\right) \quad R_2 = \max\left(\frac{|\Psi^n_{i,j} - \Psi^{n-1}_{i,j}|}{\Psi^n_{i,j}}\right)$$

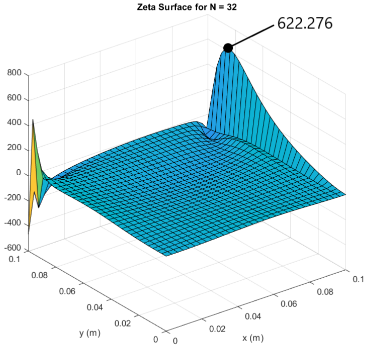
$$R_3 = \max\left(\frac{|u^n_{i,j} - u^{n-1}_{i,j}|}{u^n_{i,j}}\right) \quad R_4 = \max\left(\frac{|v^n_{i,j} - v^{n-1}_{i,j}|}{v^n_{i,j}}\right)$$

Table: Solution Residuals

N	R_1	R_2	R_3	R_4
16	1.1355e-11	1.3403e-11	3.8458e-12	2.2045e-11
32	8.1698e-4	1.2320e-4	2.2533e-4	3.5410e-4
48	5.9612e-8	1.6708e-8	4.2680e-9	6.6733e-9
64	1.6388e-8	9.4765e-9	4.2710e-9	1.4373e-9
96	1.4431e-4	1.7145e-4	4.1748e-5	4.8822e-5
128	2.9813e-4	4.9043e-5	1.3184e-3	1.9313e-4

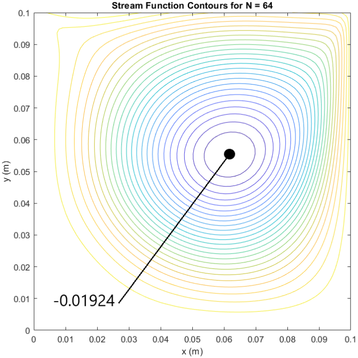
A Coarse Baseline Solution

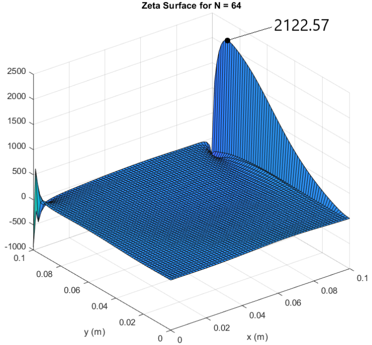




- 900 linear equations solved at each time step
- "Inverse" of a 900 square, spares matrix computed at each time step

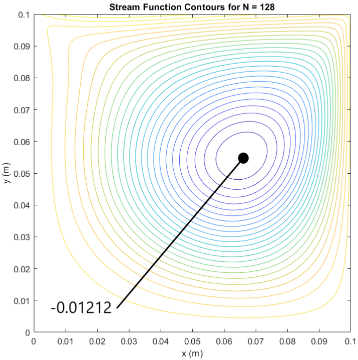
A Finer Intermediate Solution

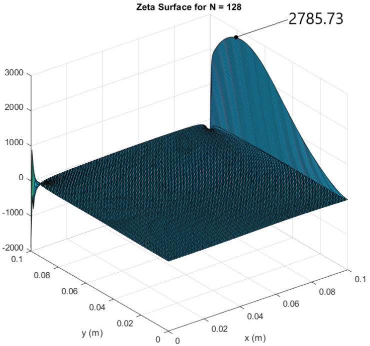




- 3844 linear equations solved at each time step
- "Inverse" of a 3844 square, spares matrix computed at each time step

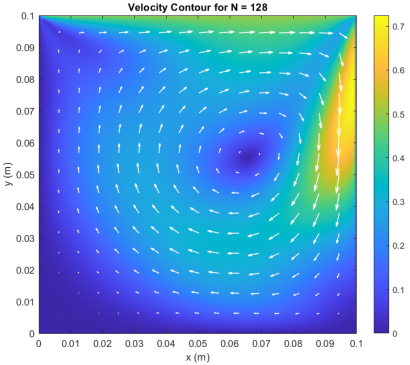
A Fine "Final" Solution

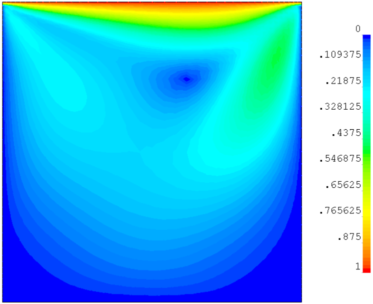




- 15876 linear equations solved at each time step
- "Inverse" of a 15876 square, spares matrix computed at each time step

Velocity Plots





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[2] http://www.nacad.ufrj.br/ rnelias/gallery/cavity.html

An Alternative Explicit Time-Stepping Method

TASE Operator Definition

For the system of first order differential equations

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = LY,$$

The p^{th} order TASE operator is recursively defined as:

$$T_L^{(p)}(\alpha, \Delta t) = \begin{cases} (I - \alpha \Delta t L)^{-1}, & \text{if } p = 1 \\ \frac{2^{p-1} T_L^{(p-1)}(\alpha/2, \Delta t) - T_L^{(p-1)}(\alpha \Delta t)}{2^{p-1} - 1}, & \text{if } p \geq 2 \end{cases},$$

where

$$\alpha \geq \alpha_{\min} = \frac{2^p - 1}{|\lambda \Delta t|_{\max}}.$$

3

[3] https://arxiv.org/abs/1911.08399

Lessons Learned

- Explicit TASE operators take longer than implicit methods when dealing with linear partial differential equations.
- Two dimensional problems yield very large matrices very quickly.
- Implementing CFD code is difficult!